Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(01)00003-X

J. Appl. Maths Mechs, Vol. 65, No. 1, pp. 13-18, 2001

0021-8928/01/$—see front matter

THE MOTION OF A THREE-LINK SYSTEM ALONG A PLANE†

F. L. CHERNOUS'KO

Moscow

(*Received* 31 *August* 2000)

Controllable motions of a three-link system along a horizontal plane when there is dry friction are considered. Previously obtained results are generalized and refined. © 2001 Elsevier Science Ltd. All rights reserved.

1. THE MECHANICAL MODEL

The biomechanics of the motion of snakes and other animals without extremities have been considered in a number of publications [1–4]. Allowance has been made for the presence of a support on channel walls [1] or vertical surfaces [2]; bends in a vertical plane have been considered [3] and models with wheels have been investigated [4]. It has been shown [5] that a simple three-link system may move in any direction along a horizontal plane only by allowing bends in the plane and assuming that there is dry friction between the system and the plane. A mode of locomotion has been proposed and the displacements and velocity of the motion have been estimated.

In this paper the same mechanical model and principle of motion as in [5] will be used. We will consider a more general control law for the motion and more general contact conditions. With these generalizations, which take into account experience in the experimental implementation of the model, conditions will be derived for the motions to be feasible. These conditions generalize and refine the results obtained in [5].

Consider a plane three-link system $O_1C_1C_2O_2$ moving along a fixed horizontal plane Oxy (see the figure). For simplicity, we will assume that the entire mass of the system is concentrated at the end points O_1 and O_2 – masses m_0 – and at the hinges C_1 and C_2 – masses m_1. The total mass of the system is $m = 2(m_0 + m_1)$. The links are assumed to be weightless rigid rods. The link C_1C_2 of length $2a$, together with the masses concentrated at the hinges C_1 and C_2, will be called the body, and the links O_1C_1 and C_2, both of length l, together with the masses at the endpoints, will be called the end links.

Let x, y denote the Cartesian coordinates of the centre of mass of the body C_1C_2, θ the angle at which the body is inclined to the x axis, and α_i the angles between the body and the end links O_iC_i $(i = 1, 2)$.

Each of the point masses O_i, C_i $(i = 1, 2)$ is subject to a dry friction force obeying Coulomb's law. The coefficient of friction for mass m_0 is k_0, and that for mass m_1 is k_1.

Control torques M_1 and M_2 are produced by motors mounted in the hinges C_1 and C_2. We shall say that the torques M_1 and M_2 are applied to the end links O_1C_1 and O_2C_2, respectively, then the body is subject to torques $-M_1$ and $-M_2$.

Any motion of the three-link system may be constructed as a combination of certain simpler motions, which we will refer to as elementary motions [5].

2. ELEMENTARY MOTIONS

Elementary motions (EMs) begin and end in a state of rest. They are characterized by the laws governing the variation of the angles $\alpha_i(t)$ $(i = 1, 2)$ of rotation of the end links relative to the body. Either one or both angles $\alpha_i(t)$ may vary in an EM. In the second case, both end links rotate synchronously, either in the same direction or in opposite directions, so that

$$\dot{\alpha}_2(t) = \pm\,\dot{\alpha}_1(t) \tag{2.1}$$

Unlike our previous approach [5], the variation of the angular velocity $\omega(t) = |\dot{\alpha}_i(t)|$ is assumed to be quite arbitrary. The only important condition is that in any EM both links begin and finish rotating simultaneously, and the angles α_i vary in the range $(-\pi, \pi)$.

†*Prikl. Mat. Mekh.* Vol. 65, No. 1, pp. 15–20, 2001.

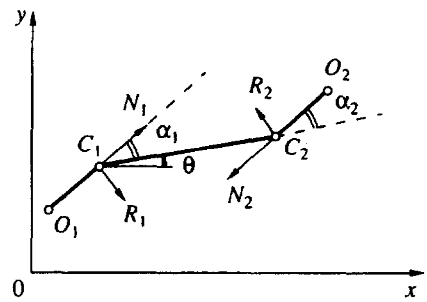

Fig. 1

We distinguish between slow and fast EMs. In slow EMs the magnitudes of the angular velocity $\omega(t)$ and angular acceleration $\varepsilon(t) = \dot{\omega}(t)$ of the end links are fairly small, so that the body C_1C_2 remains stationary. The duration of the slow EMs is denoted by T. The conditions for slow EMs to be possible will be derived below.

In fast EMs, whose duration is fairly short ($\tau \ll T$), the angular velocities $\omega(t)$ and accelerations $\varepsilon(t)$ are, conversely, fairly large. In this case the magnitudes of the control torques M_1 and M_2 are large compared with the torques created by the friction forces

$$|M_i| \gg m^* g k^* l^*, \quad m^* = \max(m_0, m_1) \tag{2.2}$$

$$k^* = \max(k_0, k_1), \quad l^* = \max(l, a)$$

When considering fast EMs, therefore, friction forces need not be taken into account. Consequently, the laws of conservation of momentum and of angular momentum hold for fast EMs.

3. CONSTRUCTION OF THE MOTIONS

We will now show how any motion of a three-link system along a horizontal plane can be built up from EMs [5]. Suppose that at the start the system is at rest with all its links parallel to the x axis. In this state we have $\theta = \alpha_1 = \alpha_2 = 0$. For brevity, we will denote slow and fast motions by the letters S and F, respectively, indicating the limits of the variation of the angles α_i in each elementary motion (from α_i^0 to α_i^1) symbolically: $\alpha_i^0 \to \alpha_i^1$. Throughout what follows, $\gamma \in (-\pi, \pi)$ will denote a certain fixed angle.

Longitudinal motion consists of the following EMs:

1) S, α_1: $0 \to \gamma$, $\alpha_2(t) \equiv 0$;
2) F, α_1: $\gamma \to 0$, α_2: $0 \to \gamma$;
3) S, α_1: $0 \to -\gamma$, α_2: $\gamma \to 0$;
4) F, α_1: $-\gamma \to 0$, α_2: $0 \to -\gamma$;
5) S, α_1: $0 \to \gamma$, α_2: $-\gamma \to 0$.

It is obvious that after step 5 the system has the same configuration as after step 1: $\alpha_1 = \gamma$, $\alpha_2 = 0$. After that, the cycle of the four EMs 2–5 may be repeated any desired number of times. For the system to reach its original straight-line configuration, $\alpha_1 = \alpha_2 = 0$, at the end of the motion, it is sufficient to perform a slow motion:

M, α_1: $\gamma \to 0$, $\alpha_2 \equiv 0$.

In the course of the slow motions, the body remains stationary, while the centre of mass of the system moves. In the course of the fast motions, conversely, the centre of mass is a fixed point and the body moves. As has already been shown [5], the total displacement of the mid-point of the body along the x axis, through the cycle of motions 2–5, is equal to

$$\Delta_0 x = 8 m_0 m^{-1} l \sin^2(\gamma/2), \quad m = 2(m_0 + m_1) \tag{3.1}$$

The total displacement of the mid-point of the body along the y axis and the total rotation of the body over a full cycle both vanish: $\Delta_0 y = 0$, $\Delta_0\theta = 0$. Since the duration of the fast motions τ is much less than that of the slow motions, it follows that the total duration of a cycle is approximately $2T$, and the mean velocity of longitudinal motion is

$$v_1 = \Delta_0 x(2T)^{-1} \tag{3.2}$$

Lateral motion is described as follows:

1) S, $\alpha_1: 0 \to -\gamma$, $\alpha_2: 0 \to \gamma$;
2) F, $\alpha_1: -\gamma \to \gamma$, $\alpha_2: \gamma \to -\gamma$;
3) S, $\alpha_1: \gamma \to -\gamma$, $\alpha_2: -\gamma \to \gamma$.

The system has the same configuration after step 3 as after step 1: $\alpha_1 = -\gamma$, $\alpha_2 = \gamma$. The cycle of two motions 2 and 3 may be repeated. In order to return to the original linear configuration $\alpha_1 = \alpha_2 = 0$, it is sufficient to perform the motion

S, $\alpha_1: -\gamma \to 0$, $\alpha_2: \gamma \to 0$.

In the cycle of motions 2 and 3, the total displacement of the mid-point of the body along the x axis and the total rotation of the body amount to zero: $\Delta_0 x = 0$, $\Delta\theta = 0$. The total displacement per cycle along the y axis and the average velocity of lateral motion are

$$\Delta_0 y = 4m_0 m^{-1} l \sin\gamma,\ v_2 = \Delta_0 y T^{-1} \tag{3.3}$$

Rotation of the system is achieved as follows:

1) S, $\alpha_1: 0 \to \gamma_1$, $\alpha_2: 0 \to \gamma_1$;
2) F, $\alpha_1: \gamma_1 \to \gamma_2$, $\alpha_2: \gamma_1 \to \gamma_2$;
3) S, $\alpha_1: \gamma_2 \to \gamma_1$, $\alpha_2: \gamma_2 \to \gamma_1$,

where γ_1 and γ_2 are angles in the range $(-\pi, \pi)$. Motions 2 and 3 may be repeated. To return the system to its original linear configuration $\alpha_1 = \alpha_2 = 0$, one has to perform the motion

S, $\alpha_1: \gamma_1 \to 0$, $\alpha_2: \gamma_1 \to 0$.

The total displacement of the mid-point of the body in the cycle of motions 2 and 3 is zero: $\Delta_0 x = \Delta_0 y = 0$; the total angle of rotation $\Delta_0\theta$ depends on the angles γ_1 and γ_2 and was determined in [5].

4. THE CONDITIONS FOR THE FEASIBILITY OF THE MOTIONS

We will now derive the sufficient conditions for the body to remain stationary during slow motions. To do this, assuming the body C_1C_2 to be stationary, we will determine the forces and torques applied to it by the rotating links O_1C_1 and O_2C_2. We will then formulate equilibrium equations for the body allowing for the interaction of the links and friction forces. The body will be stationary if friction forces at rest exist which satisfy Coulomb's law and ensure that the equilibrium equations are satisfied.

Following the scheme just outlined, we first formulate the equations of motion of the links O_iC_i ($i = 1, 2$). The equation of the torques is

$$m_0 l^2 \ddot{\alpha}_i = M_i - m_0 g k_0 l \operatorname{sign}\dot{\alpha}_i,\quad i = 1, 2 \tag{4.1}$$

Let R_i and N_i denote the components of the reaction force applied at the end of link O_iC_i by the body (see the figure). These components are determined from the equations of motion of the centre of mass of the end links. The force N_i is directed along the link O_iC_i and is equal to

$$N_i = m_0 l \dot{\alpha}_i^2 \tag{4.2}$$

while the component R_i is perpendicular to the link O_iC_i and is determined by the equations

$$R_i = m_0 l \ddot{\alpha}_i + m_0 g k_0 \operatorname{sign}\dot{\alpha}_i = M_i l^{-1} \tag{4.3}$$

The stationary body is subject to the components of the forces ($-N_i$), ($-R_i$) exerted by the end links, to torques $-M_1$ and $-M_2$, and also to friction forces at the points C_1 and C_2. Let the Ox axis of the coordinate system Oxy be directed along the body C_1C_2 and let X_i and Y_i denote the projections of the friction forces at the points C_i ($i = 1, 2$) on the x and y axes, respectively. As the three equilibrium equations of the body we take the condition that the sums of torques created by the forces applied to the body about the points C_1 and C_2 vanish, and the condition that the sum of the projections of all forces on the x axis also vanish. We obtain

$$\begin{aligned} &2a(N_2 \sin\alpha_2 - R_2\cos\alpha_2) + 2aY_2 - M_1 - M_2 = 0 \\ &2a(N_1 \sin\alpha_1 - R_1\cos\alpha_1) - 2aY_1 - M_1 - M_2 = 0 \\ &-N_1\cos\alpha_1 - R_1\sin\alpha_1 + N_2\cos\alpha_2 + R_2\sin\alpha_2 + X_1 + X_2 = 0 \end{aligned} \tag{4.4}$$

This system is statically indeterminate: we have only three equations (4.4) for four unknown forces X_i, Y_i ($i = 1, 2$). In addition, the inequalities of Coulomb's law must be satisfied

$$(X_i^2 + Y_i^2)^{1/2} \leqslant F_1, \quad F_1 = m_1 g k_1, \quad i = 1,2 \tag{4.5}$$

To achieve equilibrium, it will suffice to find at least one pair of forces X_i, Y_i ($i = 1, 2$) satisfying relations (4.4) and (4.5). The forces Y_i are uniquely defined by the first two equations of (4.4)

$$Y_i = \pm(N_i \sin\alpha_i - R_i\cos\alpha_i) \mp Q, \qquad i = 1,2 \tag{4.6}$$

where we have introduced the notation

$$Q = (M_1 + M_2)/(2a) \tag{4.7}$$

Put

$$X_i = N_i\cos\alpha_i + R_i\sin\alpha_i, \qquad i = 1,2 \tag{4.8}$$

The last equation of (4.4) is thereby satisfied. Substituting formulae (4.6) and (4.8) into the left-hand side of inequality (4.5), simplifying and using the inequality

$$|a\sin\alpha + b\cos\alpha| \leqslant (a^2 + b^2)^{1/2}$$

which holds for any a, b and α, we obtain

$$\begin{aligned} &X_i^2 + Y_i^2 = N_i^2 + R_i^2 + Q^2 - 2Q(N_i\sin\alpha_i - R_i\cos\alpha_i) \leqslant \\ &\leqslant N_i^2 + R_i^2 + Q^2 + 2|Q|(N_i^2 + R_i^2)^{1/2} = [(N_i^2 + R_i^2)^{1/2} + |Q|]^2 \end{aligned} \tag{4.9}$$

We introduce the following notation

$$\omega_0 = \max|\dot\alpha_i|, \; \varepsilon_0 = \max|\ddot\alpha_i| \tag{4.10}$$

where the maxima are evaluated over all slow motions; by condition (2.1) they are independent of $i = 1, 2$. Relations (4.1)–(4.3) and (4.10) imply the following estimates

$$|M_i| \leqslant m_0 l(l\varepsilon_0 + gk_0), \quad |R_i| \leqslant m_0(l\varepsilon_0 + gk_0), \quad |N_i| \leqslant m_0 l\omega_0^2, \quad i = 1,2 \tag{4.11}$$

If the end links rotate in the same direction, i.e., the plus sign is taken in (2.1), then $M_1 = M_2$. In that case it follows from (4.7) and (4.11) that

$$|Q| \leqslant m_0 l(l\varepsilon_0 + gk_0)/a \tag{4.12}$$

Substituting estimates (4.11), (4.12) into (4.9), we conclude that inequalities (4.5) will be satisfied provided that

$$m_0l\{[\omega_0^4+(\varepsilon_0+gk_0l^{-1})^2]^{1/2}+(\varepsilon_0+gk_0l^{-1})la^{-1}\}\leqslant m_1gk_1 \qquad (4.13)$$

If only one end link participates in the slow motion, then one of the torques M_i will vanish, estimate (4.12) holds as before and condition (4.13) is again a sufficient condition for the body to be in equilibrium.

If the end links rotate in opposite directions, that is, the minus sign is taken in (2.1), then $M_1 = -M_2$ and, by (4.7), $Q = 0$. In that case we deduce from (4.11) and (4.9) the following sufficient condition for inequalities (4.5) to hold

$$m_0l[\omega_0^4+(\varepsilon_0+gk_0l^{-1})^2]^{1/2}\leqslant m_1gk_1 \qquad (4.14)$$

In longitudinal motion and rotation of the mechanism, the end links will rotate in the same direction in slow motions, but in lateral motion they will always rotate in opposite directions (see Section 3). Hence conditions (4.13) are sufficient for longitudinal motion and rotation to be feasible, while the sufficient condition for lateral motion to be feasible is less restrictive, having the form (4.14).

Note that condition (4.13) is satisfied if the motion takes place sufficiently slowly, that is, ω_0 and ε_0 are sufficiently small and in addition

$$m_0k_0(a+l)<m_1k_1a$$

Condition (4.14) is satisfied if ω_0 and ε_0 are sufficiently small and, in addition, $m_0k_0 < m_1k_1$.

In [5] we considered a special case of slow motions in which the magnitude of the angular velocity of the end links $\omega(t)$ at first increases linearly and then decreases linearly; the angular acceleration was assumed to be constant in magnitude. In that case,

$$\omega(t)=\varepsilon_0t,\ t\in[0,T/2];\ \omega(t)=\varepsilon_0(T-t),\ t\in[T/2,T] \qquad (4.15)$$

The feasibility conditions given for this case in [5] – inequalities (4.14) and (4.16) (and conditions (6.3) and (7.3), which follow from them) – were incorrect, because formula (4.1) for R_i in [5] omitted the second term, shown in formula (4.3) of the present paper. This error has been rectified in the above conditions (4.13) and (4.14). In addition, as the bounds have been improved, the left-hand side of our present inequality (4.13) does not contain the coefficient $\sqrt{2}$, as in inequality (4.14) of [5].

As an example, let us consider a three-link system with the following parameters

$$\begin{aligned}&a=l=0.2\,\text{m},\ m_1=1.2\,\text{kg},\ m_0=0.4\,\text{kg}\\&m=3.2\ \text{kg},\ k_1=k_0=0.2,\ g=9.81\ \text{m}\cdot\text{s}^{-2}\end{aligned} \qquad (4.16)$$

Suppose the slow motions are described by Eqs (4.15). For this case,

$$\omega_0=\varepsilon_0T/2,\ \ |\Delta\alpha|=|\alpha_i^1-\alpha_i^0|=\varepsilon_0T^2/4$$

The maximum angle of rotation of the end links γ and the maximum angular acceleration ε_0 are taken to be $\gamma = 1$ rad and $\varepsilon_0 = 4$ rad·s^{-2}. According to Section 3, for longitudinal motions, $|\Delta\alpha| = \gamma$, and it follows from (4.16) that $T = 1$ s and $\omega_0 = 2$ rad·s^{-2}; for lateral motions, $|\Delta\alpha| = 2\gamma$, $T = 1.4$ s and $\omega_0 = 2.8$ rad·s^{-2}. Verification of the feasibility conditions (4.13) and (4.14) shows that they are indeed satisfied. Calculating the displacements and average velocities of the motions by formulae (3.1)–(3.3), we obtain

$$v_1=0.023\ \text{m}\cdot\text{s}^{-1},\ v_2=0.034\ \text{m}\cdot\text{s}^{-1}$$

According to estimates (2.2), the control torques necessary to realize fast motions must be an order of magnitude greater than those created by the friction forces. In this example, the torques must be of the order of 6–8 N·m.

As has been shown, conditions (4.13) and (4.14) derived above hold for fairly general laws of motion of the links and for different coefficients of friction for masses m_1 and m_0. In the experimental implementation of the proposed motion, carried out at the Munich Technical University by F. Pfeiffer, M. Gienger and G. Mayr, the control laws used were such that the angular velocity $\omega(t)$ and angular acceleration $\varepsilon(t)$ varied smoothly, while the coefficients of friction for the masses m_1 and m_0 were different. The experiments demonstrated that this mode of motion is feasible in practice.

This research was supported financially by the Russian Foundation for Basic Research (99-01-00258).

REFERENCES

1. LAVRENT'YEV, M. A. and LAVRENT'YEV, M. M., A principle for the creation of a tractive force in motion. *Zh. Prikl. Mekh. Tekh. Fiz.*, 1962, **4**, 3–9.
2. GRAY, J., *Animal Locomotion*. Weidenfeld & Nicolson, London, 1968.
3. DOBROLYUBOV, A. I., *Travelling Deformation Waves*. Nauka i Tekhnika, Minsk, 1987.
4. HIROSE, S., *Biologically Inspired Robots: Snake-like Locomotors and Manipulators*. Oxford University Press, Oxford, 1993.
5. CHERNOUS'KO, F. L., The motion of a multilink system along a horizontal plane. *Prikl. Mat. Mekh.*, 2000, **64**, 1, 8–18.

Translated by D.L.